13 Tests with one-sided null and alternative hypotheses

1. Review: α-level MP

   (1) α-level MP
   For $H_0: \theta = \theta_0$ versus $H_a: \theta = \theta_1$

   $$\phi(X) = \begin{cases} 1 & \frac{f_1}{f_0} > k \\ r & \frac{f_1}{f_0} = k \\ 0 & \frac{f_1}{f_0} < k \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)] = \alpha \text{ is an } \alpha\text{-level MP test}$$

   (2) Test in (1) is an unbiased test

   Proof. We need to show $E_{\theta_0}[\phi(X)] \le E_{\theta_1}[\phi(X)]$.
   Let $\psi(X) \equiv \alpha$. Then $E_{\theta_0}[\psi(X)] = \alpha \Longrightarrow \psi(X)$ is an α-level test.
   So the power domination $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$ is true.
   Hence $E_{\theta_0}[\phi(X)] = \alpha = E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$. □

2. Review: Two $\phi(X)$s
   Suppose $\Lambda = \frac{f_2}{f_1}$ is an increasing function of $T(X)$ for all $\theta_1 < \theta_2$.

   (1) $\phi(X)$ with non-decreasing $E_\theta[\phi(X)]$

   $$\phi(X) = \begin{cases} 1 & T(X) > c \\ r & T(X) = c \\ 0 & T(X) < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)] = \alpha$$
   has non-decreasing $E_\theta(\phi(X))$ and
   if $E_{\theta_0}[\psi(X)] \le \alpha$, then $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta > \theta_0$

   For $H_0: \theta = \theta_0$ versus $H_a: \theta > \theta_0$ the above $\phi(X)$ gives an α-level UMP test.

   (2) $\phi(X)$ with non-increasing $E_\theta[\phi(X)]$

   $$\phi(X) = \begin{cases} 1 & T(X) < c \\ r & T(X) = c \\ 0 & T(X) > \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)] = \alpha$$
   has non-increasing $E_\theta(\phi(X))$ and
   if $E_{\theta_0}[\psi(X)] \le \alpha$, then $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta < \theta_0$

   For $H_0: \theta = \theta_0$ versus $H_a: \theta < \theta_0$ the above $\phi(X)$ gives an α-level UMP test.

3. Tests with one-sided null and alternative hypotheses

   (1) For $H_0: \theta \le \theta_0$ versus $H_a: \theta > \theta_0$
   $\phi(X)$ in (1) of 2 gives an α-level UMP test

   Proof. $\phi(X)$ gives an α-level test.
   Note that $E_\theta[\phi(X)]$ is non-decreasing function of θ and $E_{\theta_0}[\phi(X)] = \alpha$.

   $$\theta \in H_0 \Longrightarrow \theta \le \theta_0 \Longrightarrow E_\theta[\phi(X)] \le E_{\theta_0}[\phi(X)] = \alpha.$$

Thus $\phi(X)$ is an α-level test.

$\phi(X)$ **is UMP test.**

Note that if $E_{\theta_0}[\psi(X)] \le E_{\theta_0}[\phi(X)]$, then $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta > \theta_0$.

If $\psi(X)$ is also an α-level test, then $E_{\theta_0}[\psi(X)] \le \alpha = E_{\theta_0}[\psi(X)]$.

Hence $E_\theta[\psi(X)] \le E_\theta[\psi(X)]$ for all $\theta > \theta_0$.

Thus $\phi(X)$ is UMP test.

Therefore $\phi(X)$ is an α-level UMP test.

(2) For $H_0 : \theta \ge \theta_0$ versus $H_a : \theta > \theta_0$

$\phi(X)$ in (2) of 2 gives an α-level UMP test

Proof. $\phi(X)$ **gives an** α**-level test.**

Note that $E_\theta[\phi(X)]$ is non-increasing function of θ and $E_{\theta_0}[\phi(X)] = \alpha$.

$$\theta \in H_0 \Longrightarrow \theta \ge \theta_0 \Longrightarrow E_\theta[\phi(X)] \le E_{\theta_0}[\phi(X)] = \alpha.$$

Thus $\phi(X)$ is an α-level test.

$\phi(X)$ **is UMP test.**

Note that if $E_{\theta_0}[\psi(X)] \le E_{\theta_0}[\phi(X)]$, then $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta < \theta_0$.

If $\psi(X)$ is also an α-level test, then $E_{\theta_0}[\psi(X)] \le \alpha = E_{\theta_0}[\psi(X)]$.

Hence $E_\theta[\psi(X)] \le E_\theta[\psi(X)]$ for all $\theta < \theta_0$.

Thus $\phi(X)$ is UMP test.

Therefore $\phi(X)$ is an α-level UMP test.

Ex: For Poisson(λ) find UMP test on $H_0 : \lambda \ge 0.4$ vs $H_a : \lambda < 0.4$ with level 0.05. Suppose $n = 10$.

Poisson(λ) has pmf $\frac{\lambda^x}{x!}e^{-\lambda}$ and likelihood function $L(\lambda) = \frac{\lambda^{\sum x_i}}{x_1!\cdots x_n!}e^{-n\lambda}$.

With $\lambda_1 < \lambda_2$, $\Lambda = \frac{L(\lambda_2)}{L(\lambda_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)^{\sum x_i} e^{n(\lambda_1 - \lambda_2)}$ is an increasing function of $T(X) = \sum_{i=1}^n X_i \sim \text{Poisson}(n\lambda)$. With $n = 10$, let $\phi(X) = \begin{cases} 1 & \sum_{i=1}^{10} < c \\ r & \sum_{i=1}^{10} = c \\ 0 & \sum_{i=1}^{10} > c \end{cases}$

$0.05 = E_{\lambda=0.4}[\phi(X)] = P(\text{Poisson}(4) < c) + r \cdot P(\text{Poisson}(4) = c)$

$r = \frac{0.05 - P(\text{Poisson}(4)<c)}{P(\text{Poisson}(4)=c)} = \frac{0.05 - P(\text{Poisson}(4)<1)}{\text{Poisson}(4)=1)} = \frac{0.05-0.01832}{0.07326} = 0.43243$

$$\phi(X) = \begin{cases} 1 & \sum_{i=1}^{10} X_i < 1 \\ 0.43243 & \sum_{i=1}^{10} X_i = 1 \\ 0 & \sum_{i=1}^{10} X_i > 1 \end{cases} \text{ is UMP test at level 0.05}$$

L14 Generalized Neyman-Pearson lemma

1. Essence of Neyman-Pearson lemma

   (1) Essence of Neyman-Pearson lemma

$$\phi(x) = \begin{cases} 1 & f(x) - g(x) > 0 \\ r & f(x) - g(x) = 0 \\ 0 & f(x) - g(x) < 0 \end{cases} \quad \text{with} \int_x \phi(x)g(x)\,dx = c.$$

   If $\int_x \psi(x)g(x)\,dx \le c$, then $\int_x \psi(x)f(x)\,dx \le \int_x \phi(x)f(x)\,dx$.

   Proof.
$$\begin{aligned} & \int_x [\phi(x) - \psi(x)]f(x)\,dx = \int_x [\phi(x) - \psi(x)][f(x) - g(x) + g(x)]\,dx \\ = & \int_{f-g>0} (1-\psi)(f-g)\,dx + \int_{f-g=0} (r-\psi)\cdot 0\,dx \\ & + \int_{f-g<0} (-\psi)(f-g)\,dx + \int_x (\phi-\psi)g\,dx \\ \ge & \ 0. \end{aligned}$$

   Comments: $g(x)$ is a tool defining a class of functions of ψ and identifying a special one ϕ in the class. $f(x)$ is used in the comparison for ϕ with all other ψ.

   (2) Equivalent one

$$\phi(x) = \begin{cases} 1 & f(x) - kg(x) > 0 \\ r & f(x) - kg(x) = 0 \\ 0 & f(x) - kg(x) < 0 \end{cases} \quad \text{with } k > 0 \text{ and} \int_x \phi(x)g(x)\,dx = c.$$

   If $\int_x \psi(x)g(x)\,dx \le c$, then $\int_x \psi(x)f(x)\,dx \le \int_x \phi(x)f(x)\,dx$.

   Proof. Conditions $\int_x \phi(x)g(x)\,dx = c \Longleftrightarrow \int_x \phi(x)kg(x)\,dx = kc$,
   and conditions $\int_x \psi(x)g(x)\,dx \le c \Longleftrightarrow \int_x \psi(x)kg(x)\,dx \le kc$.

   (3) Neyman-Pearson lemma

$$\phi(x) = \begin{cases} 1 & f_1 - kf_0 > 0 \\ r & f_1 - kf_0 = 0 \\ 0 & f_1 - kf_0 < 0 \end{cases} \quad \text{with } k > 0 \text{ and } E_{\theta_0}[\phi(X)] = \alpha.$$

   If $E_{\theta_0}[\psi(X)] \le \alpha$, then $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.

   Proof. $\int_x \phi(x)f_0(x)\,dx = E_{\theta_0}[\phi(X)] = \beta_\phi(\theta_0)$, $\int_x \psi(x)f_0(x)\,dx = E_{\theta_0}[\psi(X)] = \beta_\psi(\theta_0)$,
   $\int_x \psi(x)f_1(x)\,dx = E_{\theta_1}[\psi(X)] = \beta_\psi(\theta_1)$ and $\int_x \phi(x)f_1(x)\,dx = E_{\theta_1}[\phi(X)] = \beta_\phi(\theta_1)$.

   Comment: $f_1 - kf_0 (>=<) 0 \Longleftrightarrow \frac{f_1}{f_0} (>=<) k$.

2. Generalized Neyman-Pearson lemma

   (1) Generalized Neyman-Pearson lemma

$$\phi(x) = \begin{cases} 1 & f(x) - \sum_{i=1}^m k_i g_i(x) > 0 \\ r & f(x) - \sum_{i=1}^m k_i g_i(x) = 0 \\ 0 & f(x) - \sum_{i=1}^m k_i g_i(x) < 0 \end{cases} \quad \text{with} \int_x \phi(x) \sum_{i=1}^m k_i g_i(x)\,dx = c.$$

   If $\int_x \psi(x) \sum_{i=1}^m k_i g_i(x)\,dx \le c$, then $\int_x \psi(x)f(x)\,dx \le \int_x \phi(x)f(x)\,dx$.

(2) Equivalent form
With $k_i > 0$ for all $i = 1, ..., m$, the condition $\int_x \phi(x) \sum_{i=1}^m k_i g_i(x)\, dx = c$ can be replaced by $\int_x \phi(x) g_i(x)\, dx = c_i,\ i = 1, ..., m$ and the condition $\int_x \psi(x) \sum_{i=1}^m k_i g_i(x)\, dx \le c$ can be replaced by $\int_x \psi(x) g_i(x)\, dx \le c_i,\ i = 1, ..., m$.

3. Test classes

(1) UMP test
UMP test is based on a test class $\mathcal{C}$. ϕ is UMP test in $\mathcal{C}$ if $\phi \in \mathcal{C}$ and for $\psi \in \mathcal{C}$ $E_\theta(\psi) \le E_\theta(\phi)$ for all $\theta \in H_a$.

(2) Test classes
α-level test class: $\{0 \le \psi \le 1 : E_\theta(\psi) \le \alpha$ for all $\theta \in H_0\}$.
Conservative α-level test class: $\{\psi : E_\theta(\psi) < \alpha$ for all $\theta \in H_0\}$
Exact α-level test class: $\{\psi : E_\theta(\psi) \le \alpha$ for all $\theta \in H_0$ and $E_\theta(\psi) = \alpha$ for some $\theta \in H_0\}$
α-level similar test class: $\{\psi : E_\theta(\psi) \le \alpha\, \forall\, \theta \in H_0$ and $E_\theta(\psi) = \alpha\ \forall\, \theta$ on bounday of $H_0\}$.
Unbiased test class: $\{\psi : E_{\theta_0}(\psi) \le E_{\theta_1}(\psi)$ for all $\theta_0 \in H_0$ and all $\theta_1 \in H_a\}$.